Matched Asymptotic Solutions for Optimum Lift Controlled Atmospheric Entry

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This paper considers the problem of optimal lift control of a hypersonic lifting body during atmospheric entry for the drag coefficient, a function of the angle of attack, and the atmospheric density, an arbitrary function of altitude. The solution obtained is valid for entering the planetary atmosphere from the Keplerian region, as well as from low altitudes. The method of matched asymptotic expansions was employed, and separate expansions were derived for the Keplerian region and for the aerodynamic region, where the aerodynamic forces are dominant. Lagrange multipliers and state variables obtained in closed form for both expansions were matched in the overlap domain. A method for estimating the order of magnitude of multipliers in various regions was discussed and will be useful in applying singular perturbation methods to a wider class of optimal control problems. For unbounded control, the lift variation can be classified into four different programs depending on the terminal altitude. Results were compared with the numerical solutions obtained by the method of steepest descent. For bounded control, there exist 12 different sequences of arcs which reduce to those obtained in an earlier study as the drag coefficient becomes independent of angle of attack.

Nomenclature

Dimensional quantities

g'	= gravitational acceleration
$g'(0)$	= gravitational acceleration at $h' = 0$
h'	= altitude
m	= vehicle mass
$\bar{m}$	= mean molecular weight of the atmospheric gas
R	= mean planteary radius
$\bar{R}$	= universal gas constant
S	= effective cross section of vehicle
t'	= time of flight
T	= atmospheric temperature
β'	= $\bar{m}g/\bar{R}T$ = inverse atmospheric scale height
$\beta'(0)$	= inverse atmospheric scale height at $h' = 0$
ρ'	= atmospheric density
ρ_0'	= atmospheric density at $h' = 0$

Nondimensional quantities

a	= $1/b$
b	= $[2/(1 + C_2)]^{1/2}$
$\bar{b}$	= $[2/(1 + \bar{C}_2)]^{1/2}$
B_1,B_2,B_3,B_4,B_5	= integration constants in the outer expansions
C_1,C_2,C_3,C_4	= integration constants in the inner expansions
C_L	= $C_L(\alpha)$ = lift coefficient
C_D	$C_{DO} + \eta C_L{}^2(\alpha)$ = drag coefficient
C_{DO}	= drag coefficient at $C_L = 0$
$C_{L\text{Min}}$	= minimum C_L
$C_{L\text{Max}}$	= maximum C_L
$\bar{C}_2$	= $-C_2$
C_{11},C_{12}	= constants in the small lift expansion
C_{LN}	= the lift program in Region N
D_1	= $[C_2/(1 + C_2)]^{1/2}$
$\bar{D}_1$	= $[\bar{C}_2/(1 + \bar{C}_2)]^{1/2}$
E	= elliptical integral of the second kind
F	= elliptical integral of the first kind
g	= $g'/g'(0)$
h	= h'/R
h^*	= h/ϵ
H	= Hamiltonian
H^0	= optimal Hamiltonian
J^0	= optimal-return function
K_D	= $C_{DO}\rho(0)/2$
K_η	= $\eta\rho(0)/2$
K	= $\rho(0)/2$
k	= $1/2\eta$
l	= $(K_D/K_\eta)^{1/2}$
t	= $t'[g'(0)/R]^{1/2}$
t^*	= t/ϵ
V	= $V'/[Rg'(0)]^{1/2}$
α	= control variable or angle of attack
γ	= flight path angle is measured counterclockwise from the local horizon
γ_c	= turning angle when one switches from boundary arcs to other arcs or vice versa
γ_s	= switching angle where the control variable changes its sign
$\bar{\gamma}$	= $[\gamma_0{}^* - (C_{11} + \epsilon C_{12})]/\epsilon$
θ	= range angle
η	= a given constant in the drag coefficient
ϵ	= $1/\beta'(0)R$
ρ	= $\rho'S/m\beta'(0)$
$\rho(0)$	= the value of ρ at $h = 0$
$\lambda_h,\lambda_v,\lambda_\gamma,\lambda_\theta$	= Lagrange's multipliers
χ	= $(\pi/2 + \gamma_0{}^*)/2$
$\bar{\chi}$	= $(\pi/2 - \gamma_0{}^*)/2$
ϕ	= $\sin^{-1}(b \sin\chi)$
$\bar{\phi}$	= $\sin^{-1}(\bar{b} \sin\bar{\chi})$
β	= $\beta'/\beta'(0)$
ELA	= extremal lift arcs, i.e., arcs on which $C_L = C_{L\text{Max}}$ or $C_L = C_{L\text{Min}}$
VLA	= variable lift arcs
MaxLA	= maximum lift arcs
MinLA	= minimum lift arcs
VLAN	= variable lift arc using the control program C_{LN}
VA	= vertical arc, i.e., arcs with $\gamma = \pm\pi/2$

Superscript

*	= quantities in the inner expansions

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Subscripts

n = the $n - 1$ th term in the inner or outer expansion for $n = 0,1,2, \ldots$
i = initial conditions or conditions at the beginning of each subarc
f = terminal conditions

I. Introduction

ANALYTIC solutions of lifting re-entry into a planetary atmosphere have been discussed extensively.[1-4] Most of these solutions were derived either under the assumption of constant lift-to-drag ratio or constant lift and constant drag coefficients.

Resulting from the difficulty of obtaining analytic solutions to problems of optimal lift control of gliding vehicles, most of the available solutions for optimal lift control during re-entry are numerical. A class of approximate analytic solutions was considered by Miele[5-6] who treated optimization of the lift program of a gliding vehicle under the assumption of a shallow, smooth glide path. Recently, an additional subclass of optimum trajectory problems for hypersonic gliding vehicles[7] was solved analytically under the assumption of constant atmospheric density, gravity and drag coefficient. The above result was generalized[8] to the problem of optimal lift control of a re-entry vehicle during atmospheric entry by assuming the atmospheric density to be exponential. The assumptions[8] correspond to those used earlier[1-4] except that the assumption of constant-lift coefficient was relaxed to allow the lift to be used as a control variable. The maximum value of the lift-to-drag ratio was assumed to be 0.5 or greater as recently discussed by Tannas.[9-10]

Analytic solutions of simplified optimal re-entry problems are important from the point of view of serving as a basis for investigation of more complicated cases, as well as providing a general understanding of the structure of optimal use of lift controls. Analytic solutions also provide a better foundation for the solution of guidance problems as typical guidance laws are based on such relatively restrictive assumptions as either constant drag, constant altitude, or equilibrium glides.[9-10]

Recently, the method of matched asymptotic expansions[11-12] was applied to lifting re-entry into the atmosphere from the Keplerian region where the gravity is dominant by Willes.[13-14] The correct application of the method to this type of problem was clarified.[15] The composite solution[15] was applied to the problem of skipping entry into the atmosphere and was shown to be in excellent agreement with the exact solution obtained by numerical integration.[16]

In this paper, the method of matched asymptotic expansions is used to solve the problem of optimal lift control of a hypersonic lifting body entering the atmosphere from the Keplerian region as well as low altitudes. This is a generalization[7-8] to the case where the drag coefficient is no longer constant but varies in a manner typical of hypersonic vehicles and the atmospheric density is an arbitrary function of altitude. In addition, the assumption of constant gravity is relaxed in this study.

As was done earlier,[15] separate expansions are introduced for the outer Keplerian region and the inner aerodynamic region. The outer and inner regions are characterized by the dominance of the gravitational and aerodynamic forces, respectively. In the present case, both the inner and outer expansions can be obtained analytically in closed form and are matched in a certain overlap domain. A composite expansion which is uniformly valid everywhere is then constructed. Some preliminary comparisons of the present results with solutions obtained by the numerical method of steepest descent show very good agreement. Higher-order solutions which can easily be constructed are under study. It is anticipated that the slight discrepancy will further be reduced if higher-order solutions are calculated.

II. Problem Formulation

The governing equations for a hypersonic lifting body entering a planetary atmosphere can be expressed in terms of nondimensional variables[15] as follows

$$dV/dt = -(2\epsilon)^{-1}C_D\rho V^2 - (1+h)^{-2}\sin\gamma \tag{1}$$

$$d\gamma/dt = (2\epsilon)^{-1}C_L\rho V + [(1+h)^{-1} - (1+h)^{-2}V^{-2}]V\cos\gamma \tag{2}$$

$$d\theta/dt = V\cos\gamma/(1+h) \tag{3}$$

and

$$dh/dt = V\sin\gamma \tag{4}$$

where the atmopheric density is an arbitrary function of altitude, i.e.,

$$\rho = \rho(0)f(h/\epsilon,h) > 0 \tag{5}$$

where the arbitrary function is always positive and at least obeys

$$\lim_{\epsilon\to 0} \frac{f(h/\epsilon,h)}{\epsilon} = 0 \qquad \text{for fixed } h \tag{6}$$

for exponential atmospheric density

$$\rho = \rho(0)\exp(-h/\epsilon) \tag{7}$$

and for the isothermal atmosphere[15]

$$\rho = \rho(0)\exp[-h/\epsilon(1+h)] \tag{8}$$

the lift coefficient $C_L = C_L(\alpha)$ and the drag coefficient obeys the relation

$$C_D = C_{DO} + \eta C_L^2(\alpha) \tag{9}$$

which is typical for a hypersonic vehicle. In Eq. (9) both C_{DO} and η are arbitrary constants and the angle of attack α is the control variable. Eq. (9) has been used by many previous investigators. Following Bryson,[17] the Hamiltonian for the present system is defined by

$$H = \lambda_V[-(2\epsilon)^{-1}C_DV^2\rho - (1+h)^{-2}\sin\gamma] + \lambda_\gamma\{(2\epsilon)^{-1}C_LV\rho + [(1+h)^{-1} - (1+h)^{-2}V^{-2}]V\cos\gamma\} + \lambda_hV\sin\gamma + \lambda_\theta V\cos\gamma(1+h)^{-1} \tag{10}$$

The optimality condition

$$\partial H/\partial\alpha = -(\tfrac{1}{2})\rho V[2\eta(\lambda_vV)C_L(\alpha) - \lambda_\gamma](\partial C_L/\partial\alpha) = 0 \tag{11}$$

gives

$$C_L(\alpha) = k\lambda_\gamma/\lambda_vV \quad \text{or} \quad \partial C_L/\partial\alpha = 0 \tag{12}$$

where $k = (2\eta)^{-1}$ is a known constant.

Since the right-hand side of Eq. (10) is independent of θ, one has

$$d\lambda_\theta/dt = 0 \tag{13}$$

hence λ_θ is constant. In this paper, the problem of entering the atmosphere and reaching a given terminal altitude h_f horizontally, i.e., $\gamma_f = 0$, while minimizing the energy loss due to aerodynamic drag, i.e., maximizing the final velocity is considered. A similar problem was studied under a more restrictive assumption in an earlier investigation.[8] The present solution also corresponds to the optimal trajectory for the problem solved numerically by Speyer and Bryson[18] and was given as an example by Bryson and Ho.[17] Since the final range is not specified, one must set

$$\lambda_\theta = 0 \tag{14}$$

Thus, Eq. (3) uncouples from the rest of the system of equations.

Furthermore, since the Hamiltonian H is independent of the time t and the final time t_f is not fixed, it follows that

$$H = 0 \tag{15}$$

Equations (12) and (15) hold everywhere and to all orders because so far no perturbations or expansions have been introduced. Differential equations governing the Lagrange multipliers are easily obtained from Eq. (10) as follows:

$$d\lambda_v/dt = -\partial H/\partial v; \quad d\lambda_\gamma/dt = -\partial H/\partial \gamma \\ d\lambda_h/dt = -\partial H/\partial h \tag{16}$$

In the present case, since the method of matched asymptotic expansions is to be applied and the atmospheric entry problem is considered, it is more advantageous to use the altitude h as the independent variable. This can be accomplished by dividing Eqs. (1, 2, and 16) by Eq. (4)

$$dV/dh = -(2\epsilon)^{-1}C_D\rho V/\sin\gamma - 1/V(1+h)^2 \tag{17}$$

$$d\gamma/dh = (2\epsilon)^{-1}C_L\rho/\sin\gamma + [(1+h)^{-1} - (1+h)^{-2}V^{-2}]\cot\gamma \tag{18}$$

$$d\lambda_v/dh = -(1/V\sin\gamma)(\partial H/\partial V) \tag{19}$$

$$d\lambda_\gamma/dh = -(1/V\sin\gamma)(\partial H/\partial \gamma) \tag{20}$$

and

$$d\lambda_h/dh = -(1/V\sin\gamma)(\partial H/\partial h) \tag{21}$$

The problem reduces to solving Eqs. (10 and 17–21) subject to the boundary conditions.

$$\left.\begin{matrix} V = V_i \\ \gamma = \gamma_i \\ h = h_i \end{matrix}\right\} \text{at } t = t_i \quad \text{and} \quad \left.\begin{matrix} \gamma = 0 \\ h = h_f \\ \lambda_v = 1 \end{matrix}\right\} \text{at } t = t_f \tag{22}$$

where V_i, γ_i, h_i, t_i, and h_f are given constants.

III. Outer Expansion

The outer expansion is introduced to study the behavior of the solution in the region where the gravitational force is dominant. This expansion is introduced by repeated application of the outer limit defined as the limit as $\epsilon \to 0$ with h and the other nondimensional quantities in Eqs. (1–4) fixed. The governing equations expressed in terms of outer variables are Eqs. (10) and (17–21). Hence, to the first order, the outer limit physically corresponds to a vanishing atmosphere. Outer expansions for the state variables and Lagrange multipliers are written in the form

$$\begin{aligned} V &= V_0(h) + \epsilon V_1(h) + 0(\epsilon^2) \\ \gamma &= \gamma_0(h) + \epsilon\gamma_1(h) + 0(\epsilon^2) \\ \lambda_h &= \lambda_{h0}(h) + \epsilon\lambda_{h1}(h) + 0(\epsilon^2) \\ \lambda_v &= \lambda_{v0}(h) + \epsilon\lambda_{v1}(h) + 0(\epsilon^2) \\ \lambda_\gamma &= \lambda_{\gamma 0}(h) + \epsilon\lambda_{\gamma 1}(h) + 0(\epsilon^2) \end{aligned} \tag{23}$$

and one then obtains the following equations for γ_0 and V_0

$$dV_0/dh = -1/(1+h)^2V_0 \tag{24}$$

$$d\gamma_0/dh = [(1+h)^{-1} - (1+h)^{-2}V_0^{-2}]\cot\gamma_0 \tag{25}$$

The Hamiltonian to order unity is

$$H_0 = -\lambda_{v0}\sin\gamma_0/(1+h)^2 + \lambda_{\gamma 0}[(1+h)^{-1} - (1+h)^{-2}V_0^{-2}]V_0\cos\gamma_0 + \lambda_{h0}V_0\sin\gamma_0 \tag{26}$$

and the governing equations for the Lagrange multipliers are

$$d\lambda_{v0}/dh = -\lambda_{h0}/V_0 - \lambda_{\gamma 0}[(1+h)^{-1} + (1+h)^{-2}V_0^{-2}]V_0^{-1}\cot\gamma_0 \tag{27}$$

$$d\lambda_{\gamma 0}/dh = \lambda_{v0}\cot\gamma_0/(1+h)^2V_0 - \lambda_{h0}\cot\gamma_0 + \lambda_{\gamma 0}[(1+h)^{-1} - (1+h)^{-2}V_0^{-2}] \tag{28}$$

$$d\lambda_{h0}/dh = -2\lambda_{v0}/(1+h)^3V_0 + \lambda_{\gamma 0}[1/(1+h)^2 - 2/(1+h)^3V_0^2]\cot\gamma_0 \tag{29}$$

Solutions for Eqs. (24) and (25) are the well-known Keplerian solutions

$$V_0^2 - 2/(1+h) = B_1 \tag{30}$$

$$(1+h)V_0\cos\gamma_0 = B_2 \tag{31}$$

Equations (25) and (28) give

$$d\lambda_{\gamma 0}/d\gamma_0 = \lambda_{\gamma 0}\csc\gamma_0\sec\gamma_0 \tag{32}$$

or

$$\lambda_{\gamma 0} = B_3|\tan\gamma_0| = \pm B_3\tan\gamma_0 \tag{33}$$

where the upper or lower sign in Eq. (33) are taken to be for positive or negative γ_0, respectively. Similarly, Eqs. (26), (29), and (33) give

$$d\lambda_{h0}/dh + 2\lambda_{h0}/(1+h) = \mp B_3/(1+h)^2 \tag{34}$$

Thus

$$\lambda_{h0} = \mp B_3h/(1+h)^2 + B_4/(1+h)^2 \tag{35}$$

Finally, Eqs. (24, 26, 27, and 35) give

$$d(\lambda_v V)/dh = \mp 2B_3/(1+h)^2 - 2B_4/(1+h)^2 \tag{36}$$

or

$$\lambda_{v0} = (1/V_0)[2(B_4 \pm B_3)/(1+h) + B_5] \tag{37}$$

In the above $B_1, \ldots B_5$ are integration constants to be specified later. Substituting Eqs. (33, 35, and 37) into Eq. (26), one obtains the condition

$$B_3 - B_1(B_3 - B_4) - B_5 = 0 \tag{38}$$

Equations (30, 37, and 38) then give

$$\lambda_{v0} = (1/V_0)[B_3(1 - V_0^2) + B_4V_0^2] \tag{39}$$

Since in the above first-order solution the effect of the atmosphere is absent, the results are not valid for small h where the aerodynamic forces become dominant. The solution for the motion in this region is discussed next. In particular, certain constants of integration corresponding to $B_1, B_2, \ldots B_5$ will arise in the solution for small h inner solution and the matching conditions together with the boundary conditions on the problem will be used to specify all the undefined quantities.

It is also worth mentioning that the control variable is absent in the above first-order outer equations. However, it can be determined from Eq. (12) by using the above multipliers Eqs. (33), (35), and (39). The control has no effects to this order in the outer region. The reason is that the terms containing the control variable $C_L(\alpha)$ are multiplied by the atmospheric density which is of higher order. Physically, it means that at higher altitude, the atmospheric density is too low to provide enough lift and drag forces to affect the motion to this order. However, if initial conditions are given at higher altitude, i.e., Keplerian region, it is essential to know the complete control history along the whole trajectory including its variation in the outer region. Furthermore, multipliers and state variables in this region will affect the multipliers, state variables, the control and the structure of the control in the inner region.

IV. Inner Expansions

The inner expansion is employed to study the solutions in the region where the aerodynamic forces are dominant. The inner limit is defined as the limit $\epsilon \to 0$ while $h^* = h/\epsilon$, $t^* =$

t/ϵ, and all other physical quantities in Eqs. (1–4) are held fixed. Implicit in the definition of h^* is the fact that the inner solution is valid for relatively lower altitudes, i.e., in the region where $h = 0(\epsilon)$ above the surface of the Earth. Thus the governing equations in the inner region become

$$dV/dt^* = -(\tfrac{1}{2})C_D\rho V^2 - \epsilon \sin\gamma/(1 + \epsilon h^*)^2 \quad (40)$$

$$d\gamma/dt^* = (\tfrac{1}{2})C_L\rho V + \epsilon[1/(1 + \epsilon h^*) - 1/(1 + \epsilon h^*)^2V^2]V \cos\gamma^* \quad (41)$$

$$dh^*/dt^* = V \sin\gamma \quad (42)$$

It is easy to verify that the following scaling of the Hamiltonian and Lagrange multipliers preserves the form of the Euler-Lagrange equations, [c.f. Eq. (16)] in the starred quantities

$$H^* = \epsilon H; \quad \lambda_v^* = \lambda_v; \quad \lambda_\gamma^* = \lambda_\gamma \text{ and } \lambda_h^* = \epsilon\lambda_h \quad (43)$$

However, the requirement that the form of Euler-Lagrange equations be preserved does not yield a unique scaling as can be verified by the choice

$$H^* = H; \quad \lambda_v^* = \lambda_v\epsilon^{-1}; \quad \lambda_\gamma^* = \lambda_\gamma\epsilon^{-1} \text{ and } \lambda_h^* = \lambda_h \quad (44)$$

which also satisfies this requirement. It will be shown later that with the above scaling, none of the Lagrange multipliers match with the corresponding outer expansions. In fact, matching will uniquely require use of the scaling in Eq. (43). In essence, this scaling which establishes, a priori, the relative order of magnitude of the Lagrange multipliers in the two regions is to be justified a posteriori by matching. One of the fundamental difficulties encountered in the numerical integration of control problems such as the one formulated is Sec. II is the fact that one must solve a two-point boundary value problem. In general, no a priori estimates on the appropriate initial magnitudes exist for quantities which must satisfy the terminal constraints. This is especially true for the case of the Lagrange multipliers, i.e., the initial magnitude of λ_v is not easily established for the present problem when all that is known is that $\lambda_v = 1$ at $t = t_f$. As a result, many iterations on the unknown initial values must be made to converge on a solution satisfying the appropriate terminal values and quite often this iteration process is not convergent. Aside from providing an understanding of the qualitative nature of the control program, an important contribution of an analytic theory, approximate though it may be, is the definition of all the parameters throughout the trajectory. Thus, if need be, one can calculate with relatively few iterations an exact optimal solution based on the analytically determined initial values. If one expresses the governing equations in terms of inner variables and assumes the following expansions

$$\begin{aligned} V &= V_0^*(h^*) + \epsilon V_1^*(h^*) + O(\epsilon^2) \\ \gamma &= \gamma_0^*(h^*) + \epsilon\gamma_1^*(h^*) + O(\epsilon^2) \\ \lambda_v^* &= \lambda_{v0}^*(h^*) + \epsilon\lambda_{v1}^*(h^*) + O(\epsilon^2) \\ \lambda_\gamma^* &= \lambda_{\gamma 0}^*(h^*) + \epsilon\lambda_{\gamma 1}^*(h^*) + O(\epsilon^2) \\ \lambda_h^* &= \lambda_{h0}^*(h^*) + \epsilon\lambda_{h1}^*(h^*) + O(\epsilon^2) \end{aligned} \quad (45)$$

the equations for q_0^*, γ_0^*, λ_{v0}^*, $\lambda_{\gamma 0}^*$ and λ_{h0}^* become

$$dV_0^*/dh^* = -C_D\rho V_0^*/2 \sin\gamma_0^* \quad (46)$$

$$d\gamma_0^*/dh^* = C_L\rho/2 \sin\gamma_0^* \quad (47)$$

$$d\lambda_{v0}^*/dh^* = C_D\rho\lambda_{v0}^*/2 \sin\gamma_0^* \quad (48)$$

$$d\lambda_{\gamma 0}^*/dh^* = -\lambda_{h0}^* \cot\gamma_0^* \quad (49)$$

$$d\lambda_{h0}^*/dh^* = +(\lambda_{h0}^*/\rho)(d\rho/dh^*) \quad (50)$$

To order unity, the Hamiltonian is given as follows:

$$H_0^* = -\lambda_{v0}^*C_D\rho V_0^{*2}/2 + \lambda_{\gamma 0}^*C_L\rho V_0^*/2 + \lambda_{h0}^*V_0^* \sin\gamma_0^* = 0 \quad (51)$$

and the optimal condition becomes

$$C_L = k\lambda_{\gamma 0}^*/\lambda_{v0}^*V_0^* \quad (52)$$

For simplicity of notation, we introduce

$$K_D = C_{D0}\rho(0)/2; \quad K_\eta = \eta\rho(0)/2 \text{ and } K = \rho(0)/2 \quad (53)$$

Then Eqs. (46) and (48) give

$$\lambda_{v0}^*V_0^* = k/C_0 \quad (54)$$

where C_0 is an arbitrary constant. Therefore Eq. (52) becomes

$$C_L = C_0\lambda_{\gamma 0}^* \quad (55)$$

Further, Eq. (50) gives

$$\lambda_{h0}^* = C\rho = C\rho(0)f(h^*) = C_1f(h^*) \quad (56)$$

and Eqs. (47) and (49) give

$$\sin\gamma_0^* = -KC_0\lambda_{\gamma 0}^{*2}/2C_1 + C_2 \quad (57)$$

Thus

$$\lambda_{\gamma 0}^* = \pm[(2C_1/KC_0)(C_2 - \sin\gamma_0^*)]^{1/2} \quad (58)$$

where C_0, C_1, and C_2 are constants of integration. Substituting Eqs. (54, 56, and 57) into Eq. (51) one gets

$$C_0C_1C_2 = kK_D \quad (59)$$

Hence, Eq. (55) gives

$$C_L = \pm l(1 - \sin\gamma_0^*/C_2)^{1/2} \quad (60)$$

where

$$l^2 = K_D/K_\eta = C_{D0}/\eta > 0 \quad (61)$$

For a given terminal altitude, the constant C_2 is determined. Then the control variable is known in terms of the physical variable γ_0^*. Thus, Eq. (60) is in itself, an explicit guidance law.

Then Eqs. (46, 47, and 60) give

$$\ln V_0^* = \mp(l/2k)[\int(1 - \sin\gamma_0^*/C_2)^{-1/2}d\gamma_0^* + \int(1 - \sin\gamma_0^*/C_2)^{1/2}d\gamma_0^*] + C_3 \quad (62)$$

Similarly substituting Eq. (60) into Eq. (47) gives

$$-\int f(h^*)dh^* = \mp(l/2kK_D)\int(1 - \sin\gamma_0^*/C_2)^{-1/2} \times \sin\gamma_0^*d\gamma_0^* + C_4 \quad (63)$$

One observes that Eqs. (62) and (63) can be expressed in terms of elliptic functions and can be integrated in a form depending on the value of C_2 as follows

A) $C_2 > 0$: If $C_2 > 0$ one can write

$$(1 - \sin\gamma_0^*/C_2)^{1/2} = (1/D_1)(1 - b^2 \sin^2\chi)^{1/2} \quad (64)$$

where

$$b^2 = 2/(1 + C_2) > 0; \quad \chi = (\pi/2 + \gamma_0^*)/2 \text{ and } D_1 = [C_2/(1 + C_2)]^{1/2} \quad (65)$$

There are three subcases depending on $b < 1$, $b = 1$ and $b > 1$.

1) $b < 1$ and $C_2 > 1$: In this case, we have

$$V_0^* = V_{0i}^* \exp\{\mp(l/k)[D_1F(\chi,b) + (1/D_1)E(\chi,b)]\}\Big|_{\chi_i}^{\chi} \quad (66)$$

$$g(h^*) = g(h_i^*) \mp (C_2/klK_\eta)[D_1F(\chi,b) - (1/D_1)E(\chi,b)]\Big|_{\chi_i}^{\chi} \quad (67)$$

where F is the elliptic integral of the first kind[19] with amplitude χ and modulus b and E is the elliptic integral of the second kind and the function $g(h^*)$ is

$$g(h^*) = -\int_0^{h^*} f(h^*)dh^* - f(0) \quad (68)$$

2) $b = 1$ and $C_2 = 1$: In this case, one has

$$V_0^* = V_{0i}^* \exp\{\mp(l/(2)^{1/2}k) \times [\ln(\sec\chi + \tan\chi) + 2\sin\chi]\}|_{\chi_i}^{\chi} \quad (69)$$

$$g(h^*) = g(h_i^*) \mp (1/(2)^{1/2}klK_\eta) \times [\ln(\sec\chi + \tan\chi) - 2\sin\chi]|_{\chi_i}^{\chi} \quad (70)$$

3) $b > 1$ and $C_2 < 1$: Let $a = 1/b$; $\sin\phi = b\sin\chi$. Then the solutions become

$$V_0^* = V_{0i}^* \exp\{\mp[l/(2C_2)^{1/2}k][(2C_2 - 1)F(\phi,a) + 2E(\phi,a)]\}|_{\phi_i}^{\phi} \quad (71)$$

$$g(h^*) = g(h_i^*) \mp [(C_2)^{1/2}/(2)^{1/2}klK_\eta][F(\phi,a) - 2E(\phi,a)]|_{\phi_i}^{\phi} \quad (72)$$

B) $C_2 < 0$: In this case, let $\bar{C}_2 = -C_2 > 0$ and then it follows that

$$(1 - \sin\gamma_0^*/C_2)^{1/2} = (1/\bar{D}_1)(1 - \bar{b}^2\sin^2\bar{\chi})^{1/2} \quad (73)$$

where

$$\bar{D}_1 = [\bar{C}_2/(1 + \bar{C}_2)]^{1/2};\quad \bar{b}^2 = 2/(1 + \bar{C}_2);\quad \bar{\chi} = (\pi/2 - \gamma_0^*)/2 \quad (74)$$

Again, there are three subcases depending on $\bar{b} < 1$, $\bar{b} = 1$ and $\bar{b} > 1$. 1) $\bar{b} < 1$ and $\bar{C}_2 > 1$:

$$V_0^* = V_{0i}^* \exp\{\pm(l/k)[\bar{D}_1F(\bar{\chi},\bar{b}) + (1/\bar{D}_1)E(\bar{\chi},\bar{b})]\}|_{\bar{\chi}}^{\bar{\chi}} \quad (75)$$

$$g(h^*) = g(h_i^*) \pm (\bar{C}_2/klK_\eta)[\bar{D}_1F(\bar{\chi},\bar{b}) - (1/\bar{D}_1)E(\bar{\chi},\bar{b})]|_{\bar{\chi}}^{\bar{\chi}} \quad (76)$$

2) $\bar{b} = 1$ and $\bar{C}_2 = 1$:

$$V_0^* = V_{0i}^* \exp\{\pm[l/(2)^{1/2}k][\ln(\sec\bar{\chi} + \tan\bar{\chi}) + 2\sin\bar{\chi}]\}|_{\bar{\chi}_i}^{\bar{\chi}} \quad (77)$$

$$g(h^*) = g(h_i^*) \pm [1/(2)^{1/2}klK_\eta][\ln(\sec\bar{\chi} + \tan\bar{\chi}) - 2\sin\bar{\chi}]_{\bar{\chi}_i}^{\bar{\chi}} \quad (78)$$

3) $\bar{b} > 1$ and $\bar{C}_2 < 1$: Let $\bar{a} = 1/\bar{b} < 1$; $\sin\phi = \bar{b}\sin\bar{\chi}$. Then

$$V_0^* = V_{0i}^* \exp\{\pm[l/(2\bar{C}_2)^{1/2}k][(2\bar{C}_2 - 1)F(\bar{\phi},\bar{a}) + 2E(\bar{\phi},\bar{a})]\}|_{\bar{\phi}_i}^{\bar{\phi}} \quad (79)$$

$$g(h^*) = g(h_i^*) \pm [(\bar{C}_2)^{1/2}/(2)^{1/2}klK_\eta][F(\bar{\phi},a) - 2E(\bar{\phi},\bar{a})]|_{\bar{\phi}_i}^{\bar{\phi}} \quad (80)$$

In the above, the subscript i denotes initial conditions if only inner solutions are needed and denotes constants to be determined by matching with the outer expansions if the solution starts in the outer region. For an exponential atmospheric density, one can simply substitute

$$g(h^*) = \exp(-h^*) \quad (81)$$

in the above equations.

V. Matching and Composite Solutions

Matching between the physical variables is essentially the same as given before.[15] Without going into detail,[11–12] one can apply the following simple matching condition

$$V_0(0) = V_0^*(\infty) = B_1 + 2 = V_{oo} \quad (82)$$

and

$$\gamma_0(0) = \gamma_0^*(\infty) = \cos^{-1}[B_2/(B_2 + 2)^{1/2}] = \gamma_{oo} \quad (83)$$

where V_{oo} and γ_{oo} correspond to the value of the outer expansions V_0 and γ_0 at $h = 0$, respectively.

Comparing the expression

$$\lambda_{v0}V_0 = -2(B_3 - B_4)/(1 + h) + B_5 = -2(B_3 - B_4) + B_5 \text{ as } h \to 0 \quad (84)$$

derived from the outer expansion with the result

$$\lambda_{v0}^*V_0^* = k/C_0 = \text{const} \quad (85)$$

for the inner expansion leads to

$$-2(B_3 - B_4) + B_5 = k/C_0 \quad (86)$$

Similarly

$$B_3 = (l/C_0\tan\gamma_{oo})[1 - \sin\gamma_{oo}/C_2]^{1/2}$$
$$B_4 = k/V_{oo}^2C_0 - B_3(1 - V_{oo}^2)/V_{oo} \quad (87)$$
$$B_5 = B_3(1 + B_1) - B_1B_4$$

Thus, matching determines the constants of the outer expansion in terms of those of the inner expansion. A systematic way of determining all constants and solutions will be discussed later.

Consider next matching between λ_h and λ_h^*. The outer expansion gives

$$\lambda_{h0} = \mp B_3h/(1 + h)^2 + B_4/(1 + h)^2 = B_4 \text{ as } h \to 0 \quad (88)$$

The inner expansion gives $\lambda_{h0}^* = C_1f(h^*)$ and for an exponential atmospheric density, one has

$$\lambda_{h0}^* = C_1\exp(-h^*) \to 0 \text{ as } h^* \to \infty \quad (89)$$

Therefore, matching requires $\lambda_{h0}^* = \epsilon\lambda_{h0}$. On the other hand, if the scaling of Eq. (44) is taken, matching gives

$$B_4 = 0 \quad (90)$$

This together with the remaining matching conditions imply that all the λ's are equal to zero. In fact, it is easy to verify that the only scaling which leads to nontrival λ's is the one adopted. This illustrates, for the present problem, the earlier statements regarding the use of matching to determine the appropriate order of magnitude for the multipliers.

Following the usual method of constructing uniformly valid asymptotic solutions, i.e., composite solutions, one obtains:

$$\gamma_c = \gamma_0 + \gamma_0^* - \gamma_{oo} + 0(\epsilon) \quad (91)$$

$$V_c = V_0 + V_0^* - V_{oo} + 0(\epsilon) \quad (92)$$

$$\lambda_{vc} = (1/V_0)\{-2(B_3 - B_4)/(1 + h) + B_5\} + k/C_0V_0^* - (1/V_{oo})\{-2(B_3 - B_4) + B_5\} + 0(\epsilon) \quad (93)$$

$$\lambda_{\gamma c} = \pm B_3\tan\gamma_0 \pm (l/C_0)(1 - \sin\gamma_0^*/C_2)^{1/2} \mp B_3\tan\gamma_{oo} + 0(\epsilon) \quad (94)$$

and

$$\lambda_{hc} = \lambda_{h0} + (1/\epsilon)\lambda_{h0}^* + 0(\epsilon) \quad (95)$$

The above composite solutions are uniformly valid to order unity for all value of h and this completes the solution to order unity. A complete history of control variable can be determined from Eq. (12) by using the above composite multipliers. The above composite solutions can be used to satisfy boundary conditions in the Keplerian region as well as aerodynamic region.

VI. Discussion of Inner Solutions

In order to understand how the solutions for the complete problem with bounded control can be constructed systematically and the proper use of various optimal lift programs, it is useful and interesting to study the inner solutions in detail. However, in the inner region, the composite solutions and the inner solution differ only in higher order terms.

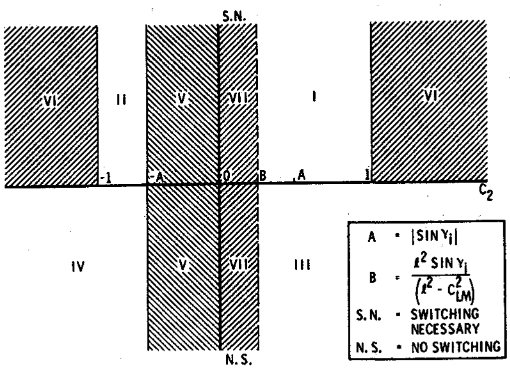

Fig. 1 Lift programs vs C_2.

Furthermore, the control is most effective only in the inner region and if the initial altitude of entry is not high, most problems can be treated adequately with the inner solution alone. Therefore, in order to give a simple and clear picture of the structure of control programs and how boundary arcs are inserted if needed, only the inner solutions need be discussed in detail. Generalization to the case of using composite solutions is straightforward.

From Eq. (60), one can see that the lift force can only change sign at $C_L = 0$, i.e., at the point

$$\sin\gamma_0{}^* = \sin\gamma_s = C_2 \tag{96}$$

where γ_s is the flight-path angle at the switching point, i.e., the point where the lift changes sign. If the lift changes sign at any other point, the Erdmann-Weierstrass corner conditions are obviously violated.

Since the value of C_2 directly determined the lifting force and since the most important topic for the present problem is the application of lift control, it is advantageous to classify solutions into various lift programs according to the value of C_2.

Since h_i', h_f', γ_i and $\gamma_f = 0$ are known, one can assume a value for C_2 and calculate the corresponding h_f' from Eq. (63), and iterate until the calculated terminal altitude h_{fc} agrees with the given terminal altitude h_f. In practice, this iteration converges rapidly. For one of the more complicated examples discussed, it only took seven iterations to obtain $h_{fc}' = 249{,}744$ ft instead of the given terminal altitude $h_f' = 250{,}000$ ft.

One can observe from Eq. (60) and Fig. 1, that for terminal altitudes of practical interest, there are four types of lifting programs for maneuvering entry. In Fig. 1, the abscissa represents the value of C_2. Above this line, the regions are divided according to the value of C_2 and the assumption that there exists at least one switching point in the lift program. Below this line, the regions are divided according to the value of C_2 and the assumption that there exists no switching point. The present discussion is restricted to the case where the initial flight-path angle γ_i satisfies $-\pi/2 < \gamma_i < 0$. This covers almost all the cases of practical interest. In Region I, the lift program is started with a positive lift arc and is switched to a negative lift arc at $\gamma = \gamma_s > 0$. In Region II, the lift program starts with a negative lift arc followed by a positive lift arc with the switching point at $\gamma = \gamma_s$ where $\gamma_s < \gamma_i$. In Region III, the lift is always positive and decreases from its maximum at the initial point to its minimum at the terminal point. However, this will be restricted by the boundedness of the control. Thus, for a given bound, boundary arcs may have to be used for a certain terminal altitude and this case will be discussed later. In Region IV, the lift is always positive and increases from its minimum at the initial point to its maximum at the terminal point. In Region V, there are no solutions because the lift becomes imaginary. In Region VI, there exists no solutions because there exists no switching point. In Region VII, the calculated lift is greater than the maximum lift obtainable for bounded control. This region exists separately from Region I and III, if boundary arcs are needed. The lift programs in each region are sketched in Fig. 2.

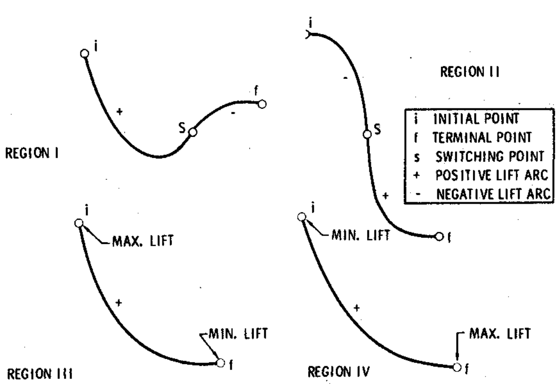

Fig. 2 Sketch of lift programs in regions I, II, III, and IV.

VII. Numerical Examples

In this section, the above discussion is applied to some numerical examples. In order to see how the four lift programs can be used to reach all the terminal altitudes of interests the boundedness of the control is not imposed. An exponential atmospheric density is assumed. The case considered has the following lift and drag coefficients

$$C_L(\alpha) = C_{L0}\alpha \quad \text{and} \quad C_D(\alpha) = C_{D0} + C_{L0}\alpha^2 \tag{97}$$

The problem is solved for the following initial conditions

$$V_i' = 36{,}000 \text{ fps}; \quad \gamma_i = -7.5° \text{ and } h_i' = 400{,}000 \text{ ft} \tag{98}$$

for $C_{L0} = 0.9$, $C_{D0} = 0.3$, $\beta'^{-1} = 23{,}500$ ft, $m = 3.8$ slugs, $S = 3.985$ ft^2 and $\rho_0' = 0.0027$ slugs/ft^3 and first for the following terminal conditions

$$\gamma_f = 0 \quad \text{and} \quad h_f' = 250{,}000 \text{ ft} \tag{99}$$

It can easily be shown that the case considered belongs to Region I and is the most complicated case because there exists a switching point. The flight-path angle changes from negative to positive values and passes through zero before the lifting body reaches its final horizontal position. The solution is a trajectory composed of a positive lift arc followed by a negative lift arc. The switching point where the lift switches from negative to positive or vice versa is defined by Eq. (96) with $\sin\gamma_s = C_2 > 0$. Using Eq. (72), one can now estimate a value of C_2, determine the values of elliptic integrals E and F from ϕ_0 to ϕ_f and solve for h_f. Repeating this procedure, one can eventually determine the value of C_2 which will give the desired h_f.

For the present case, $C_2 = 0.042915$ and Eq. (62) can then be used to calculate the terminal velocity

$$V_{0f}{}^* = 1.0236 \tag{100}$$

and the remaining constants are determined as follows

$$C_0 = k/V_{0f}{}^* = 0.43962 \text{ and } C_1 = kK_D/C_0C_2 = 238.064 \tag{101}$$

The initial and terminal values of the Lagrange multiplier are determined as follows:

$$\begin{aligned} &\lambda_{v0}{}^*(t_i{}^*) = k/C_0V_{0i}{}^* = 0.73772; \quad \lambda_{v0}{}^*(t_f{}^*) = 1 \\ &\lambda_{\gamma 0}{}^*(t_i{}^*) = 2.37619; \quad \lambda_{\gamma 0}{}^*(t_f{}^*) = 1.1896 \\ &\lambda_{h0}{}^*(t_i{}^*) = 0.96482 \times 10^{-5}; \\ &\qquad \lambda_{h0}{}^*(t_f{}^*) = 0.55969 \times 10^{-2} \\ &\alpha(t_i{}^*) = 66.50°; \quad \alpha(t_f{}^*) = -33.08° \end{aligned} \tag{102}$$

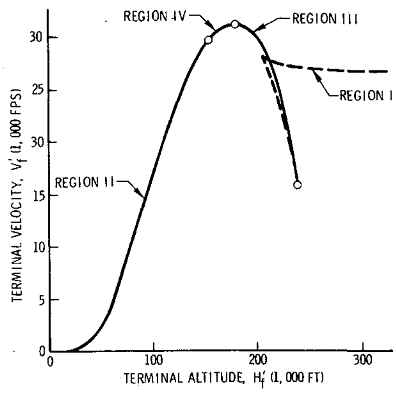

Fig. 3 Terminal altitude vs terminal velocity.

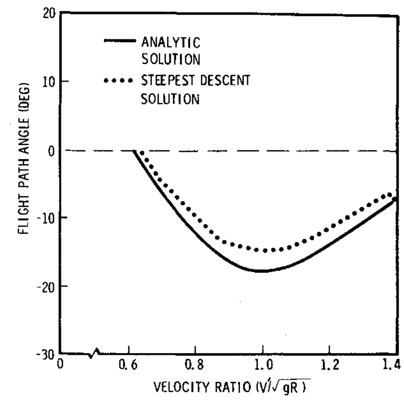

Fig. 5 Flight-path angle vs velocity ratio: example A.

Another interesting aspect is that although the Lagrange multipliers may be of order 10^5 in dimensional quantities, they are of order unity or small in nondimensional quantities as shown here. This is important from the point of view of applying the singular perturbation method to actual numerical examples.

From numerical results, the terminal altitude can be plotted against the value of C_2. It can be shown that the four lift programs do cover all the terminal altitudes of interest as expected. For mission planning purposes, one can easily find from such plot what kind of lift programs are needed to reach a given terminal altitude. The terminal altitude vs the terminal velocity is plotted in Fig. 3. This plot will give a rough idea about what the terminal velocity will be if one wishes to reach a certain terminal altitude. This is important from the point of view of mission planning if one plans to make further maneuvering from this position. Furthermore, such a plot can immediately show that for a given terminal altitude, how many sequences of arcs can reach it and what is the real extremal one. In order to test the accuracy of our inner solutions, the following examples were solved in detail and are compared with the numerical solutions obtained by the method of steepest descent.

Example A

The same initial conditions as Eq. (98) but with different terminal conditions, i.e., $\gamma_f = 0°$ and $h_f' = 100{,}000$ ft. This terminal altitude can be reached by use of the lift program in Region II, as shown in Fig. 2.

Example B

The same terminal conditions as Eq. (99) but with different initial conditions, i.e., $V_i' = 30{,}125$ fps, $h_i' = 191{,}530$ ft and $\gamma_i' = 0°$. This corresponds to the optimal trajectory considered by Speyer and Bryson.[17] The same problem was also briefly discussed as an example by Bryson and Ho.[18] According to their terminology, this corresponds to control of a lifting re-entry vehicle at super circular velocity from the bottom of the pullup maneuver to a higher altitude at zero flight-path angle while maximizing the terminal velocity. Comparison of the above examples and the numerical solutions are shown in Figs. 4–7.

It should be pointed out that the numerical solutions obtained by the method of steepest descent are based on a different atmospheric density model, 1959 ARDC Atmosphere. Thus, some of the discrepancy may be caused by the difference in atmospheric density. Furthermore, some of the discrepancy at higher altitudes such as shown in Fig. 5 may be caused by the errors of the method of steepest descent since the control is less effective for the present problem at such higher altitudes. The discrepancy may also be attributed to the fact that the lifting body is no longer in the inner region. The use of composite expansions instead of inner expansions may improve the comparison.

In spite of the above minor discrepancies, the comparisons in Figs. 4–7 show surprisingly good agreement in view of the fact that the numerical solution itself may have considerable errors. For instance, Shi and Eckstein[20] recently obtained an exact analytic solution for a rather simple lunar soft landing problem formulated and solved numerically by Teng and Kumar.[21] Comparison of the exact analytic solution and the extensive numerical solutions reported by Teng and Kumar[21] show that the error for the 16 cases considered ranges from less than 1% to as large as 15%.

VIII. Discussion of Solutions in the Small Lift Region

In the inner region, aerodynamic forces are assumed to be greater than gravitational forces. Therefore, a brief discussion of how to obtain solutions in the region where the lifting force vanishes in Eq. (41) is necessary. When C_L becomes $0(\epsilon)$, Eq. (41) shows that the lift term becomes the same order of magnitude as the gravity term. One has to consider the effect of the gravity term in the $\gamma_0{}^*$-equation. On the other hand, the term proportional to $C_L{}^2$ in the drag equation, c.f. Eq. (40) can be neglected to order ϵ. The question to be in-

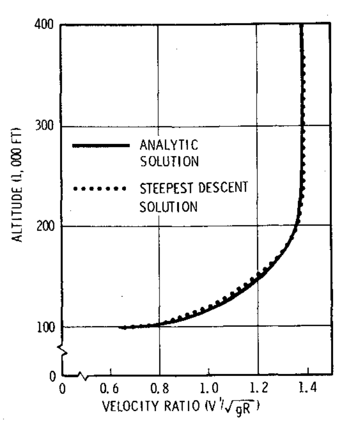

Fig. 4 Altitude vs velocity ratio: example A.

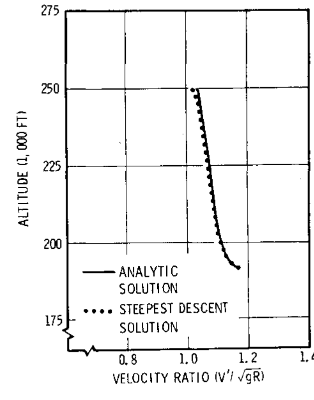

Fig. 6 Altitude vs velocity ratio: example B.

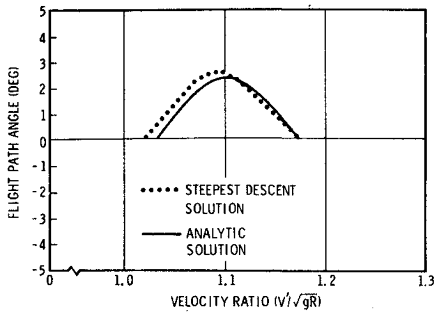

Fig. 7 Flight-path angle vs velocity ratio: example B.

vestigated here is whether in the region where $C_L = 0(\epsilon)$, another expansion is needed. However, one should remember that C_L is not always $0(\epsilon)$ and the lifting body passes through $C_L = 0(\epsilon)$ from $C_L = 0(1)$ and only on VLAI and VLAII. One can simply let $C_L^* = C_L/\epsilon$ and investigate the solutions in this region.

1. Almost Vertical Arcs

The first investigation is whether the trajectory can stay in the region $C_L = 0(\epsilon)$ for $t^* = 0(1)$ or $h^* = 0(1)$. From Eq. (60), one can see that for $C_L = 0(\epsilon)$, the variation of $\gamma_0{}^*$ is small. One can introduce a new variable

$$\bar{\gamma} = [\gamma_0{}^* - (C_{11} + \epsilon C_{12})]/\epsilon \tag{103}$$

where C_{11} and C_{12} are constants to be determined, and investigate all possible solutions in this region. The conclusion is that the only possible solution is for $C_{11} = \pm\pi/2$. This arc is almost vertical; i.e., vertical with possible variation of order ϵ; otherwise, the solution is not optimal. One can essentially conclude here that there exists no optimal arcs for which $C_L = 0(\epsilon)$ and $t^* = 0(1)$ or $h^* = 0(1)$ unless $\gamma_0{}^*$ approaches $\pm\pi/2$ as $\epsilon \to 0$. One can further show that solutions in this region are essentially contained in the inner solutions for VLAI and VLAII with C_2 approaches ± 1. This is also physically obvious. Although $C_L = 0(\epsilon)$ in this region, the gravity in the γ^*-equation is of even higher order because it is multiplied by $\cos\gamma^*$ which vanishes as γ^* approaches $\pm\pi/2$.

2. Arcs of Infinitesimal Length

In addition to the aforementioned arcs, there may exist regions where $C_L = 0(\epsilon)$ and $t^* = 0(\epsilon^\beta)$ with $\beta > 0$ or $h^* = 0(\epsilon^\beta)$; i.e., the lifting body passing through this small region in a very short time. The effects of the gravity term can be estimated separately for $C_2 = 0(1)$ and for small C_2. However, it can be shown that they have no effect to solutions here. The numerical example B shows that there is very little discrepancy for $C_2 = 0.04279$.

3. Vertical Arcs

One can easily show that the following solution

$$C_L = \lambda_\gamma = 0 \quad \text{and} \quad \gamma = \pm\pi/2 \tag{104}$$

satisfies the exact equations to all order. These arcs are exactly vertical.

IX. Boundary Arcs and Boundedness of the Control

In the preceding discussion, the boundedness of the control was not imposed in order to give a simple and clear picture of the structure of various lift control programs. It is to be shown here that once the case of unbounded control is understood, boundary arcs can easily be inserted and sequences of arcs can easily be determined. If the boundedness of the control is given as follows

$$C_{L\text{Min}} \leq C_L \leq C_{L\text{Max}} \quad \text{or} \quad \alpha_{\text{Min}} \leq \alpha \leq \alpha_{\text{Max}} \tag{105}$$

and if for the simplification of the following discussion, we assume that

$$C_{L\text{Min}} = -C_{L\text{Max}} = -C_{LM} \text{ or } \alpha_{\text{Min}} = -\alpha_{\text{Max}} = -\alpha_M \tag{106}$$

then there may exist boundary arcs on which

$$C_L = C_{LM} \quad \text{or} \quad C_L = -C_{LM} \tag{107}$$

It is necessary to have $\partial H/\partial\alpha > 0$ on Max LA and $\partial H/\partial\alpha < 0$ on Min LA. On these boundary arcs, the control is constant and is equal to its upper or lower bound. For the present problem, boundary arcs are needed if the condition

$$-C_{LM} < C_L(\alpha) < C_{LM} \tag{108}$$

is violated except at the boundary point.

For the sake of understanding how boundary arcs are inserted, let us first consider the case

$$-C_{LM} \leq C_L(t_f{}^*) \leq C_{LM} \tag{109}$$

For this case, it can be seen that the lift program II and IV are unchanged and no boundary arcs are needed for C_2 in Regions II and IV because Eq. (108) is not violated. As for the lift programs I and III, they have to be modified. As C_2 approaches zero, Eq. (108) is obviously violated no matter what the magnitude of the bound for the control is. Suppose for $C_2 = B$, one has

$$C_L(t_i{}^*) = l(1 - \sin\gamma_{0i}{}^*/B)^{1/2} = C_{LM} \tag{110}$$

For those terminal altitudes previously reached by $C_2 < B$ for the case of unbounded controls, the lift Programs I and III have to be modified because boundary arcs are to be inserted. Since Eq. (110) holds, the sequence of arcs is obviously MaxLA followed by the VLA and for a given C_2 the switching from MaxLA to VLA is then at

$$\sin\gamma_c = C_2(l^2 - C_{LM}{}^2)/l^2 \tag{111}$$

for both lift Programs I and III. Thus, for a given C_2, the lift program is completely determined. Of course, the region where $C_2 < B$ has to be modified for the case of bounded control. Equation (111) shows that the turning angle γ_c (we denote γ_c as the turning angle where one switches from boundary arcs to VLA and vice versa) depends on C_2 only for a given problem. Since for a given C_2, the sequence of arcs and γ_c is known, one could use the iteration scheme similar to the one used for the case of unbounded control to determine the desired value of C_2 for a given h_f.

In the following, all possible sequences of arcs for all possible values of the boundedness on the control are summarized according to C_2 in various regions in Fig. 1. In region N where N = 1,2,3,4,... or I,II,III,IV,..., the lift program N is used on the variable lift arcs, i.e., VLA, which for simplicity are denoted by VLAN and the corresponding lift coefficients are denoted by C_{LN}. Similarly MaxLA, MinLA, and ELA are abbreviated for maximum lift arc, minimum lift arc and extremal lift arcs respectively in the following discussions.

One can easily show that for a given C_2, the value of γ_s and γ_c are determined from Eqs. (96) and (111). Therefore, the solution is known, if the sequence of arcs is determined. All possible sequences of arcs in Regions I, II, III, and IV are summarized in Table 1, 2, 3, and 4, according to the value of C_{LM}.

In order to investigate whether it is possible to have other sequence of arcs or jumps in the control variable. It is necessary to study solutions on extremal lift arcs (ELA) and vertical arcs (VA). By requiring that $\lambda_{v0}{}^*$, $\lambda_{h0}{}^*$, and H^* be continuous at corners, i.e., Erdmann-Weierstrass corner condi-

Table 1 Sequence of arcs in region I

Path	C_{LM}	Sequence of arcs	γ_c
a	$C_{LM} \geq C_{L1}(t_i^*) > l$	VLAI	None
b	$C_{L1}(t_i^*) > C_{LM} \geq l$	MaxLA to VLAI	$\gamma_s > 0$ $\gamma_c < 0$
c	$C_{L1}(t_i^*) > l > C_{LM}$	MaxLA to VLAI to MinLA	$\gamma_s < \pi/2$ $\gamma_s > \gamma_c > 0$

tions, solutions on ELA and VA are summarized as follows:
1) solutions on ELA

$$\lambda_{v0}^* V_0^* = k/C_0 \quad \text{and} \quad \lambda_{h0}^* = C_1 f(h^*) \tag{112}$$

$$\lambda_{\gamma 0}^* = (kK_D/C_0K_{LM}^*)[(l^2 + C_{LM}^2)/l^2 - (C_0C_1/kK_D)\sin\gamma_0^*] \tag{113}$$

and

$$V_0^* = V_{0i}^* \exp\{-(K_D + K_\eta C_{LM}^2) \times (\gamma_0^* - \gamma_{0i}^*)/K_{LM}^*\} \tag{114}$$

$$\cos\gamma_0^* = K_{LM}^*[g(h_i^*) - g(h^*)] + \cos\gamma_{0i}^* \tag{115}$$

where $K_{LM}^* = K_{LM}$ for MaxLA and $K_{LM}^* = -K_{LM}$ for MinLA.
2) Solutions on VA:

$$C_L = \lambda_\gamma = 0 \quad \text{and} \quad \gamma = \pm\pi/2 \tag{116}$$

$$\lambda_{v0}^* V_0^* = k/C_0 \quad \text{and} \quad \lambda_{h0}^* = C_1 f(h^*) \tag{117}$$

$$\ln(V_0^*/V_{0i}^*) = \pm K_D[g(h^*) - g(h_i^*)] \tag{118}$$

Since the Erdmann-Weierstrass corner conditions further require that $\lambda_{\gamma 0}^*$ be continuous at corners. One can easily show that ELA can only be connected to VLA at $\gamma_0^* = \gamma_c$, and it is impossible to connect ELA to VA unless $\eta = 0$. Thus, VA is no use in the present case. In addition, all other possible sequences such as replacing part of the VLA by ELA, using two VLA with different C_2 on the same sequence or using VA are ruled out because the Erdmann-Weierstrass corner conditions are violated.

X. Conclusion and Discussion

An interesting aspect of the present problem is that various questions about putting various types of arcs together in proper sequences for the case of singular control in Ref. 8 are quite simple and obvious here. In the limit $\eta \to 0$, $C_L(\alpha)$ becomes infinite everywhere on VLAIII and VLAIV. Thus, for a given bound, paths III a, b, c and IV a, b, c become a maximum lift arc. For VLAI and VLAII, $|C_L|$ is greater than C_{LM} everywhere except in a small region (in γ_0^* — variable) around the switching point where $\sin\gamma_s = C_2$ or $\gamma_s = \sin^{-1}C_2$ as η becomes small. Boundary arcs are used everywhere except this region. In the limit $\eta \to 0$, the length of this region where C_L varies from C_{LM} to $-C_{LM}$ or vice versa, vanishes if $|C_2| < 1$. Paths I a, b, c with $C_2 < 1$ become the sequence of a maximum lift arc followed by a minimum lift arc. Paths II a, b, c with $C_2 > -1$ become the sequence of a minimum lift arc followed by a maximum lift arc. The control becomes the so-called bang-bang type. However, for VLAI and VLAII with C_2 very close to ± 1, the small region where the lift varies, becomes VA of finite length as $\eta \to 0$. In order to see how this is approached in the limit, let us first consider the case $C_2 \to -1$. In this case, the small region is $-\pi/2 < \gamma_s \leq \gamma_0^* \leq \gamma_c$ where Eq. (111) gives $\gamma_c = \sin^{-1}[-1 + 0(\eta)]$ which becomes $-\pi/2$ as $\eta \to 0$. The VLAI is used in this region and therefore becomes VA in the limit as $\eta \to 0$. The length of this VA depends on how close is C_2 to -1. Without going into the details, one can show that if

$$C_2 = -1 + \delta(\eta) \tag{119}$$

the length, i.e., $(h_i - h_f)$ of VA may be obtained from the following equation

$$g(h_f^*) - g(h_i^*) = [4(2)^{1/2}/\rho(0)C_{Do}]C \tag{120}$$

which is the limit of Eq. (80) as $\eta \to 0$ for

$$\delta(\eta) = \exp\{-2C/(\eta)^{1/2}\} \quad \text{and} \quad C > 0 \tag{121}$$

where the arbitrary constant C indicates how close is C_2 to -1. The relation between V_0^* and h^* reduces to that of Eq. (118).

Thus in the limit as $\eta \to 0$, VLAII with $C_2 = -1 + \delta(\eta)$ becomes VA of finite length and paths II a, b, c with $C_2 = -1 + \delta(\eta)$ become the sequence of a minimum lift arc followed by a vertical arc to a maximum lift arc. One can also show that the control becomes gang-bang or discontinuous at the beginning and at the end of this limiting VA. Similarly, in the limit as $\eta \to 0$, path I a, b, c with $C_2 = 1 - \delta(\eta)$ becomes the sequence of a maximum lift arc followed by a vertical arc to a minimum lift arc. In the limit as $\eta \to 0$, these sequence are essentially the solutions for the constant drag coefficient problem.[8]

From our solution, one can also see how the control is actually switching from one arc to another arc continuously but becomes discontinuous or bang-bang as $\eta \to 0$. The situation is similar to that of passing through a shock wave in a viscous fluid as compared to a shock discontinuity in an inviscid fluid. In that case, the small quantity is the kinematic viscosity ν and in the limit $\nu \to 0$, inviscid fluid, the physical quantities such as velocity, pressure, etc., change discontinuously from one side of shock wave to the other side and the shock wave becomes a line of discontinuity with zero thickness. In the present case, by studying this nonlinear control problem (the problem is nonlinear in physical variables as well as the control variable), it helps to understand how the limiting problem of the bang-bang control is achieved.

In addition, the present solution can be applied to a much wider class of hypersonic lifting bodies because the assumption of constant drag coefficient is not made. Furthermore, application of the method of matched asymptotic expansion will ensure that the higher order solutions can be system-

Table 2 Sequence of arcs in region II

Path	C_{LM}	Sequence of arcs	γ_c
a	$\|C_{L2}(t_i^*)\| < l \leq C_{LM}$	VLAII	$\gamma_s < 0$
b	$\|C_{L2}(t_i^*)\| \leq C_{LM} < l$	VLAII to MaxLA	$\gamma_s < 0$ $\gamma_c > \gamma_{0i}^* > \gamma_s$
c	$C_{LM} < \|C_{L2}(t_i^*)\| < l$	MinLA to VLAII to MaxLA	$\gamma_s < 0$ $\gamma_{0i}^* > \gamma_c > \gamma_s$

Table 3 Sequence of arcs in region III

Path	C_{LM}	Sequence of arcs	γ_c
a	$l < C_{L3}(t_i^*) \leq C_{LM}$	VLAIII	None
b	$l < C_{LM} < C_{L3}(t_i^*)$	MaxLA to VLAIII	$\gamma_{0i}^* < \gamma_c < 0$
c	$C_{LM} \leq l < C_{L3}(t_i^*)$	MaxLA	None

Table 4 Sequence of arcs in region IV

Path	C_{LM}	Sequence of arcs	γ_c
a	$C_{L4}(t_i^*) < l \leq C_{LM}$	VLAIV	None
b	$C_{L4}(t_i^*) < C_{LM} < l$	VLAIV to MaxLA	$\gamma_{0i}^* < \gamma_c < 0$
c	$C_{LM} \leq C_{L4}(t_i^*) < l$	MaxLA	None

atically calculated to increase the accuracy if so desired. In addition, explicit analytic guidance laws can be formulated by using these matched asymptotic solutions.

Finally, the method developed here for estimating the relative orders of magnitude of Lagrange Multipliers in various regions of expansions may be used in applying the singular perturbation method to many other problems in optimal control. We would also like to clarify some questions concerning matching between λ_{h0} and λ_{h0}^*. Since the inner expansion for λ_{h0}^* decays exponentially for an exponential atmospheric density, one may wonder whether an expansion intermediate to the outer expansion and the inner expansion is needed. A similar situation which occurs in fluid mechanics was first discussed by Bush.[22] Lee and Cheng[23] carried out some detailed analytic and numerical studies to show that the intermediate expansion is not necessary. The present case is very similar to that of Lee and Cheng. Our inner λ_{h0}^* corresponds to the temperature T in their inner expansion of hypersonic strong-interaction similarity solutions.

In conclusion, it can be shown that the uncertainty in the proper scaling of the Langrange multipliers and the Hamiltonian in the inner region can also be clarified by using the so-called Hamilton-Jacobi-Bellman Equation, [c.f. Eqs. (14) and (15) in Sec. 4.2 of Ref. 17] used in dynamic programing.

$$H^0 = -\partial J^0/\partial t \text{ and } \lambda_i = \partial J^0/\partial x_i \begin{pmatrix} x_i = V,\gamma,h \\ \lambda_i = \lambda_v,\lambda_\gamma,\lambda_h \end{pmatrix} \quad (122)$$

where J^0 is the optimal return function and H^0 is the optimal Hamiltonian. In the present case, we are maximizing the final velocity which remains in the hypersonic range, i.e., of order unity. Thus, if one introduces the inner variables $h^* = h/\epsilon$ and $t^* = t/\epsilon$, Eq. (43) obviously gives the proper scaling if one wishes to preserve the form of the above Hamilton-Jacobi-Bellman equations in the inner region. On the other hand, it seems that if one wishes to study the minimum time problem, Eq. (44) would give the proper scaling. Thus, it is interesting to note that the variation or change of order of magnitude of the Lagrange multipliers from one region to another can be estimated either by the discussion here or by the principle of matching.

It is worth mentioning that the results presented here are unchanged if one used the variable $s = \cos\gamma$ instead of γ as in Ref. 15. The necessary condition

$$\partial^2 H/\partial\alpha^2 = -\eta\rho V(\lambda_v V)(\partial C_L/\partial\alpha)^2 \leq 0 \quad (123)$$

is also satisfied everywhere. Another interesting feature is that the control programs and the structure of the control are independent of what kind of atmospheric density model is chosen. Of course, the constant C_2, for a given terminal altitude will be slightly different for each model of atmospheric density. This paper is a shortened version of Ref. 24. The reader is referred to Ref. 24 for details and other information, which has been omitted because of space limitations.

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